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REAL POLYNOMIAL REPRESENTATIONS OF MULTI-VALUED LOGIC

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I. INTRODUCTION

This dissertation develops real polynomial representations of functions of multi-valued discrete variables. A multi-valued discrete variable is one which can take on only a finite number of discrete values. Application of the real polynomials is made to networks containing, for the most part, ternary devices.

One of the advantages of the real polynomials when analyzing networks with multi-valued logic is that they follow the usual rules of algebraic manipulation without special conventions. In addition, they are useful for approximation in the least squares best fit sense, are useful in describing weighted and non-weighted codes, are useful in describing functional decoding, and are useful in interpolation.

Other types of algebras with special conventions have been developed (6). A modular algebra has been discussed by Bernstein (1). Algebras referred to as Post algebras in the literature were initiated by Post (8). Hanson (3) presents an algebra for analyzing a ternary device.

Binary devices are widely used in the engineering art. Boolean algebra has been well developed for handling networks of binary devices. Recently, Sander (12) has developed a real polynomial algebra for handling the logic associated with binary devices.

Though not as widely used, devices exhibiting more than two discrete states do exist (2, 4, 6, 7, 11, 13). Perhaps, with the inventive genius of engineers and scientists at work, more such devices will be invented. The state of a multi-state device may be different voltage levels, different current levels, different phases of some signal with respect to a reference signal, or a combination of the preceding. The real polynomials developed in this dissertation are useful in describing the logic associated with multi-state devices.

II. REAL POLYNOMIALS OF p-ARY VARIABLES

A. Arbitrary Functions

Definition 1:

A p-valued variable is a variable x_j that can take on only one p finite real values x_j^1, x_j^2, ..., x_j^p where p is an integer greater than zero and where $x_j^m \neq x_j^n$ when $m \neq n$.

Definition 2:

A complete function of n multi-valued discrete variables where each variable is a p-valued variable, but p is not necessarily the same for each variable, is a function defined for all possible combinations of values of the n variables. An incomplete function is a function of multi-valued discrete variables that is not complete.

Observe that a complete function of two two-valued variables and one three-valued variable must be defined for the twelve possible combinations of the three variables.

Any function of multi-valued discrete variables can be represented by a finite table listing the possible combinations of values that the variables x_j take on and the value of the function for each point. An example of such a table for a complete function of two two-valued variables and one three-valued variable is shown in Table 1.

Table 1. General function of two two-valued variables and one three-valued variable

x_3	x_2	x_1	$f(x_1, x_2, x_3)$
x_3^1	x_2^1	x_1^1	y_1

Table 1 (Continued)

x_3	x_2	x_1	$f(x_1, x_2, x_3)$
x_3^1	x_2^1	x_1^2	y_2
x_3^1	x_2^2	x_1^1	y_3
x_3^1	x_2^2	x_1^2	y_4
x_3^2	x_2^1	x_1^1	y_5
x_3^2	x_2^1	x_1^2	y_6
x_3^2	x_2^2	x_1^1	y_7
x_3^2	x_2^2	x_1^2	y_8
x_3^3	x_2^1	x_1^1	y_9
x_3^3	x_2^1	x_1^2	y_{10}
x_3^3	x_2^2	x_1^1	y_{11}
x_3^3	x_2^2	x_1^2	y_{12}

Definition 3:

A set of p-ary variables is a set $x_1, x_2, \ldots, x_n$ of p-valued variables such that $x_1^1 = x_2^1 = \ldots = x_n^1$, $x_1^2 = x_2^2 = \ldots = x_n^2, \ldots,$ $x_1^p = x_2^p = \ldots = x_n^p$.

Definition 4:

The variable z_j is a p-valued variable such that $z_j^1 = 0$, $z_j^2 = 1$, $z_j^3 = 2, \ldots, z_j^p = p - 1$.

It follows directly that a set of variables z_j is a set of p-ary variables.

Definition 5:

The variable v_j is a two-valued variable such that $v_j^1 = -1$ and

$v_j^2 = +1$.

It follows directly that a set of variables v_j is a set of binary variables.

Definition 6:

The variable t_j is a three-valued variable such that $t_j^1 = -1$, $t_j^2=0$, and $t_j^3= +1$.

It follows directly that a set of variables t_j is a set of ternary variables.

Clearly, the following relation exists between a three-valued variable z_j and the t_j variable

$$z_j - 1 = t_j \text{ .} \qquad (1)$$

A function of two three-valued z variables is shown in Table 2.

Table 2. General function of two three-valued z variables

z_1	z_2	$f(z_1, z_2)$
0	0	y_1
0	1	y_2
0	2	y_3
1	0	y_4
1	1	y_5
1	2	y_6
2	0	y_7
2	1	y_8
2	2	y_9

We now proceed with a theorem which allows us to express functions by means of a real polynomial in z_j directly.

Theorem 1: Given any complete function f of two three-valued variables, z_1 and z_2, such as shown in Table 2, this function can be expressed as the following real polynomial.

$$\begin{aligned} f(z_1, z_2) = {} & y_1 (1-z_2)(2-z_2)(1-z_1)(2-z_1)\tfrac{1}{4} \\ & + y_2 (1-z_2)(2-z_2)(z_1)(2-z_1)\tfrac{1}{2} \\ & + y_3 (1-z_2)(2-z_2)(z_1)(z_1-1)\tfrac{1}{4} \\ & + y_4 (z_2)(2-z_2)(1-z_1)(2-z_1)\tfrac{1}{2} \\ & + y_5 (z_2)(2-z_2)(z_1)(2-z_1) \\ & + y_6 (z_2)(2-z_2)(z_1)(z_1-1)\tfrac{1}{2} \\ & + y_7 (z_2)(z_2-1)(1-z_1)(2-z_1)\tfrac{1}{4} \\ & + y_8 (z_2)(z_2-1)(z_1)(2-z_1)\tfrac{1}{2} \\ & + y_9 (z_2)(z_2-1)(z_1)(z_1-1)\tfrac{1}{4} \end{aligned} \qquad (2)$$

Proof: Substitution of the values of z_1 and z_2 from the first row of the function table, Table 2, yields

$$\begin{aligned} f(0,0) = {} & y_1(1) + y_2(0) + y_3(0) + y_4(0) + y_5(0) + y_6(0) + y_7(0) \\ & + y_8(0) + y_9(0) = y_1 \end{aligned} \qquad (3)$$

Similarly, substitution of the values of z_1 and z_2 from the K-th row of the function table gives

$$\begin{aligned} f(z_{1K}, z_{2K}) = {} & y_1(0) + \ldots + y_{K-1}(0) + y_K(1) \\ & + y_{K+1}(0) + \ldots + y_9(0) = y_K \end{aligned} \qquad (4)$$

Thus, the polynomial of Equation 2 has been shown to satisfy the requirements of the function table.

Theorem 1 is easily generalized to functions of multi-valued discrete variables. The procedure is to write the polynomial in the form

$$f(x_1,\ x_2,\ \ldots,\ x_n) = y_1h_1 + y_2h_2 + y_3h_3 + \ldots \tag{5}$$

where substitution of the values of the set x_j from the K-th row of the function table causes $h_K = 1$ and $h_j = 0$ where $j \neq K$. In order to illustrate the concept further, consider the function of Table 1. This function may be represented by the following polynomial.

$$\begin{aligned}
f(x_1,\ x_2,\ x_3) = {} & y_1 \frac{(x_1-x_1^2)(x_2-x_2^2)(x_3-x_3^2)(x_3-x_3^3)}{(x_1^1-x_1^2)(x_2^1-x_2^2)(x_3^1-x_3^2)(x_3^1-x_3^3)} \\
& + y_2 \frac{(x_1-x_1^1)(x_2-x_2^2)(x_3-x_3^2)(x_3-x_3^3)}{(x_1^2-x_1^1)(x_2^1-x_2^2)(x_3^1-x_3^2)(x_3^1-x_3^3)} \\
& + y_3 \frac{(x_1-x_1^2)(x_2-x_2^1)(x_3-x_3^2)(x_3-x_3^3)}{(x_1^1-x_1^2)(x_2^2-x_2^1)(x_3^1-x_3^2)(x_3^1-x_3^3)} \\
& + y_4 \frac{(x_1-x_1^1)(x_2-x_2^1)(x_3-x_3^2)(x_3-x_3^3)}{(x_1^2-x_1^1)(x_2^2-x_2^1)(x_3^1-x_3^2)(x_3-x_3^3)} \\
& + y_5 \frac{(x_1-x_1^2)(x_2-x_2^2)(x_3-x_3^1)(x_3-x_3^3)}{(x_1^1-x_1^2)(x_2^1-x_2^2)(x_3^2-x_3^1)(x_3^2-x_3^3)} \\
& + y_6 \frac{(x_1-x_1^1)(x_2-x_2^2)(x_3-x_3^1)(x_3-x_3^3)}{(x_1^2-x_1^1)(x_2^1-x_2^2)(x_3^2-x_3^1)(x_3^2-x_3^3)} \\
& + y_7 \frac{(x_1-x_1^2)(x_2-x_2^1)(x_3-x_3^1)(x_3-x_3^3)}{(x_1^1-x_1^2)(x_2^2-x_2^1)(x_3^2-x_3^1)(x_3^2-x_3^3)} \\
& + y_8 \frac{(x_1-x_1^1)(x_2-x_2^1)(x_3-x_3^1)(x_3-x_3^3)}{(x_1^2-x_1^1)(x_2^2-x_2^1)(x_3^2-x_3^1)(x_3^2-x_3^3)} \\
& + y_9 \frac{(x_1-x_1^2)(x_2-x_2^2)(x_3-x_3^1)(x_3-x_3^2)}{(x_1^1-x_1^2)(x_2^1-x_2^2)(x_3^3-x_3^1)(x_3^3-x_3^1)}
\end{aligned}$$

$$+ y_{10} \frac{(x_1-x_1^1)(x_2-x_2^2)(x_3-x_3^1)(x_3-x_3^2)}{(x_1^2-x_1^1)(x_2^1-x_2^2)(x_3^3-x_3^1)(x_3^3-x_3^2)}$$

$$+ y_{11} \frac{(x_1-x_1^2)(x_2-x_2^1)(x_3-x_3^1)(x_3-x_3^2)}{(x_1^1-x_1^2)(x_2^2-x_2^1)(x_3^3-x_3^1)(x_3^3-x_3^2)}$$

$$+ y_{12} \frac{(x_1-x_1^1)(x_2-x_2^1)(x_3-x_3^1)(x_3-x_3^2)}{(x_1^2-x_1^1)(x_2^2-x_2^1)(x_3^3-x_3^1)(x_3^3-x_3^2)} \qquad (6)$$

Substitution of values of x_1, x_2, and x_3 from the K-th row of Table 1 yields

$$f(x_{1K}, x_{2K}, x_{3K}) = y_K \qquad (7)$$

which shows the correctness of the Polynomial 6.

B. Change of Variables

Theorem 2: If x_j is a p-valued variable and r_j is another p-valued variable, the following relation exists between x_j and r_j:

$$r_j = r_j^1 + \frac{x_j-x_j^1}{x_j^2-x_j^1} (r_j^2-r_j^1 + \frac{x_j-x_j^2}{x_j^3-x_j^2} (\frac{x_j^2-x_j^1}{x_j^3-x_j^1} (r_j^3-r_j^1)-(r_j^2-r_j^1) + \frac{x_j-x_j^3}{x_j^4-x_j^3} (\frac{(x_j^2-x_j^1)}{(x_j^4-x_j^1)} \frac{(x_j^3-x_j^2)}{(x_j^4-x_j^2)} (r_j^4-r_j^1) - \frac{x_j^3-x_j^2}{x_j^4-x_j^2} (r_j^2 - r_j^1) - (\frac{x_j^2-x_j^1}{x_j^3-x_j^1} (r_j^3-r_j^1) - (r_j^2-r_j^1)) + \ldots))) \qquad (8)$$

The proof follows directly since substitution of x_j and x_j^K in Equation 8 gives $r_j = r_j^K$.

If x_j is a two-valued variable and r_j is another two-valued variable, Equation 8 becomes

$$r_j = r_j^1 + \frac{x_j-x_j^1}{x_j^2-x_j^1} (r_j^2-r_j^1) \qquad (9)$$

If x_j is a 3-valued variable and r_j is another 3-valued variable, Equation 8 becomes

$$r_j = r_j^1 + \frac{x_j - x_j^1}{x_j^2 - x_j^1} \left(r_j^2 - r_j^1 + \frac{x_j - x_j^2}{x_j^3 - x_j^2} \left(\frac{x_j^2 - x_j^1}{x_j^3 - x_j^1} (r_j^3 - r_j^1) - (r_j^2 - r_j^1)\right)\right) \qquad (10)$$

It is seen that the relation (8) is not, in general, linear between x_j and r_j.

C. Orthogonal Variables

Consider the function of Table 2 which has the polynomial representation given by Equation 2. Examination of Equation 2 shows that another form of the function is

$$f(z_1, z_2) = a_1 a_0 + a_2 z_1 + a_3 z_2 + a_4 z_1^2 + a_5 z_2^2 + a_6 z_1 z_2 + a_7 z_1^2 + a_8 z_1 z_2^2 + a_9 z_1^2 z_2^2 \qquad (11)$$

where the a's are constants determined by the y's of Table 2 and a_0 is not zero. Equation 1 shows that a linear relationship exists between the three-valued z_j and t_j so that $f(z_1, z_2)$ may be expressed as a function $g(t_1, t_2)$ as follows.

$$\begin{aligned} f(z_1, z_2) &= g(t_1, t_2) \\ &= K_1 K_0 + K_2 t_1 + K_3 t_2 + K_4 t_1 t_2 + K_5 t_1^2 + K_6 t_2^2 + K_7 t_1^2 t_2 + K_8 t_1 t_2^2 \\ &\quad + K_9 t_1^2 t_2^2 \end{aligned} \qquad (12)$$

where the K's are constants determined by the y's of Table 2 and K_0 is not zero.

There are nine terms in either Equation 11 or 12. If we let m_i denote the i-th term of either Equation 11 or 12 ($i=1, 2, \ldots, 9$), neither Equation 11 nor 12 possesses the orthogonal property that

$$\sum_{K=1}^{9} m_{iK}m_{jK} = 0 \qquad i \neq j \tag{13}$$

where K is an index on the rows of Table 2 and m_{iK} is the i-th term of either Equation 11 or 12 evaluated for the values of the variables from the K-th row of Table 2.

It is possible to generate a nine term polynomial in the variables t_j (or z_j) representing $g(t_1, t_2)$ whose terms satisfy the orthogonal relation of Equation 13. The terms of Equation 12 with the constants K_i ($i = 1, 2, \ldots, 9$) ignored are K_0, t_1, t_2, t_1^2, t_2^2, t_1t_2, $t_1^2t_2$, $t_1t_2^2$ and $t_1^2t_2^2$ where K_0 is not zero.

Let

$$q_1 = K_0 \tag{14}$$

Let

$$q_2 = d_{21}q_1 + t_1 \tag{15}$$

where d_{21} is a constant.

If q_1 and q_2 are to be orthogonal, then we must choose the constant d_{21} of Equation 15 in such a way that

$$\sum_{K=1}^{9} q_{1K}q_{2K} = 0 \tag{16}$$

Then, substituting Equation 15 in Equation 16 gives

$$\sum_{K=1}^{9} d_{21}\, q_{1K}^2 + \sum_{K=1}^{9} t_{1K}q_{1K} = 0$$

$$9d_{21}K_0^2 + K_0 \sum_{K=1}^{9} t_{1K} = 0$$

$$d_{21} = 0 \tag{17}$$

Set $d_{21} = 0$. This causes q_1 and q_2 to be orthogonal.

Next, let

$$q_3 = d_{31}q_1 + d_{32}q_2 + t_2 \tag{18}$$

where d_{31} and d_{32} are constants.

If q_1 and q_3 are to be orthogonal, then

$$\sum_{K=1}^{9} q_{3K}\, q_{1K} = 0 \quad . \tag{19}$$

Equation 19 may be used to find d_{31} since q_1 and q_2 are known to be orthogonal. Substituting Equation 18 in Equation 19 gives $d_{31} = 0$. Set $d_{31} = 0$. This causes q_1 and q_3 to be orthogonal.

Similarly, set $d_{32} = 0$ since this causes

$$\sum_{K=1}^{9} q_{3K}q_{2K} = 0 \tag{20}$$

and q_2 and q_3 are then orthogonal.

The process may be continued by letting

$$q_4 = d_{41}q_1 + d_{42}q_2 + d_{43}q_3 + t_1^{\,2} \tag{21}$$

where d_{41}, d_{42}, and d_{43} are constants.

Find d_{41} from

$$\sum_{K=1}^{9} q_{4K}q_{1K} = 0 \tag{22}$$

Find d_{42} from

$$\sum_{K=1}^{9} q_{4K}q_{2K} = 0 \tag{23}$$

Find d_{43} from

$$\sum_{K=1}^{9} q_{4K}q_{3K} = 0 \tag{24}$$

Then q_4 will be orthogonal to q_1, to q_2, and to q_3.

The process thus far indicated is continued to find q_5, q_6, q_7, q_8, q_9. The results are

$$q_1 = K_o \tag{14}$$

$$q_2 = t_1 \tag{25}$$

$$q_3 = t_2 \tag{26}$$

$$q_4 = t_1^2 - \frac{2}{3} \tag{27}$$

$$q_5 = t_2^2 - \frac{2}{3} \tag{28}$$

$$q_6 = t_1 t_2 \tag{29}$$

$$q_7 = (t_1^2 - \frac{2}{3})\, t_2 \tag{30}$$

$$q_8 = t_1\, (t_2^2 - \frac{2}{3}) \tag{31}$$

$$q_9 = (t_1^2 - \frac{2}{3})(t_2^2 - \frac{2}{3}) \tag{32}$$

where, for i = 1, 2, ..., 9, and j = 1, 2, ..., 9,

$$\sum_{K=1}^{9} q_{iK} q_{jK} = 0 \qquad i \neq j \tag{33}$$

Next, let, for i = 1, 2, ..., 9,

$$p_i = \frac{q_i}{\sqrt{\sum_{K=1}^{9} q_{iK}^2}} \quad . \tag{34}$$

Then, for i = 1, 2, ..., 9 and j = 1, 2, ..., 9,

$$\sum_{K=1}^{9} p_{iK} p_{jK} = 0 \qquad i \neq j \tag{35}$$

and

$$\sum_{K=1}^{9} p_{iK}^2 = 1 \quad . \tag{36}$$

Equation 12 may be written as

$$g(t_1, t_2) = c_1p_1 + c_2p_2 + c_3p_3 + c_4p_4 + c_5p_5 + c_6p_6 + c_7p_7 + c_8p_8 + c_9p_9 \tag{37}$$

where the c's are constants determined by the y's of Table 2. The following procedure may be used to find the c's. From Table 2,

$$y_K = c_1p_{1K} + c_2p_{2K} + c_3p_{3K} + c_4p_{4K} + c_5p_{5K} + c_6p_{6K} + c_7p_{7K} + c_8p_{8K} + c_9p_{9K} \tag{38}$$

where K is an index on the rows of Table 2 and p_{iK} $(i = 1, 2, \ldots, 9)$ is p_i evaluated for the values of the variables from the K-th row of the function table. Then, multiplying Equation 38 by p_{iK} and summing over the rows of the function table yields

$$\sum_{K=1}^{9} y_K \, p_{iK} = \sum_{m=1}^{9} \sum_{K=1}^{9} c_m p_{mK} p_{iK} \tag{39}$$

Using Equations 34 and 35 gives

$$c_i = \sum_{K=1}^{9} y_K \, p_{iK} \quad . \tag{40}$$

At this point, a summary of the above ideas is in order. A function of n multi-valued variables may be written, as indicated in Theorem 1 or as in Equation 6. This complete function may then be written in a form like that of Equation 11 or Equation 12. The terms of the latter form do not, in general, satisfy the orthogonality relation indicated by Equation 13. Using the procedures indicated from Equation 14 through Equation 24, orthogonal terms may be developed which are designated as q_i. Next employ Equation 34 to find p_i. Then, if N is the number of rows of the function table defining the function and K is an index on the rows of the function table

$$\sum_{K=1}^{N} p_{iK}p_{jK} = 0 \qquad i \neq j \tag{41}$$

for i = 1, 2, ..., N and j = 1, 2, ..., N. Also

$$\sum_{K=1}^{N} p_{iK}^{2} = 1. \tag{42}$$

The function $f\ (x_1, x_2, \ldots, x_n)$ may then be expressed as

$$f\ (x_1, x_2, \ldots, x_n) = \sum_{m=1}^{T} c_m p_m \tag{43}$$

where T is the number of terms in the polynomial representing $f(x_1, x_2, \ldots, x_n)$.

Let y_k denote the value of $f(x_1, x_2, \ldots, x_n)$ for the values of the variables from the Kth row of the function table.

$$y_K = f\ (x_{1K}, x_{2K}, \ldots, x_{nK}) \tag{44}$$

Then

$$y_K = \sum_{m=1}^{T} c_m p_{mK} \tag{45}$$

The c's may be found by multiplying Equation 45 by p_{iK}, summing over the N rows of the function table, and interchanging the order of the summation.

$$\sum_{K=1}^{N} y_K p_{iK} = \sum_{K=1}^{N} \sum_{m=1}^{T} c_m p_{mK} p_{iK} = \sum_{m=1}^{T} c_m \sum_{K=1}^{N} p_{mK} p_{iK} \tag{46}$$

Using Equations 41 and 42 gives

$$c_i = \sum_{K=1}^{N} y_K p_{iK} \tag{47}$$

If the orthogonal p_i are generated, the polynomial representing the function may be found by using Equations 43 and 47. This may be more convenient than using the method indicated by Theorem 1 to find the polynomial.

It is shown by Sander (12) that the variable v_j of Definition 5 leads to a polynomial whose terms are orthogonal for a function of any number of two-valued variables. This makes v_j of great convenience in dealing with functions of two-valued variables. As shown by Equation 9, a linear relation exists between v_j and any other two-valued variable x_j.

The variable t_j of Definition 6 is a useful three-valued variable since it does lead to simplifications in finding orthogonal terms of the p_i type. Equation 1 shows that a linear relation exists between a three-valued z_j and t_j. However, Equation 10 shows that a linear relationship does not, in general, exist between t_j and any other three-valued variable x_j. If working with three-valued variables other than z_j or t_j where no linear relation exists between the variables and z_j or t_j, it is suggested that orthogonal terms of the p_i type be generated in terms of the three-valued variables. This suggestion is extended to functions of multi-valued discrete variables. The suggestion is felt advantageous over working with z_j (or t_j) and making a nonlinear transformation back to the variables of interest.

It is noted that a complete function of two two-valued z variables can be written as

$$f(z_1, z_2) = c_1 p_1 + c_2 p_2 + c_3 p_3 + c_4 p_4 \tag{48}$$

where the c's are constants and the p's, which satisfy Equations 41 and 42, are given by

$$p_1 = \tfrac{1}{2} \tag{49}$$

$$p_2 = z_1 - \tfrac{1}{2} \tag{50}$$

$$p_3 = z_2 - \tfrac{1}{2} \tag{51}$$

$$p_4 = 2\ (z_1 z_2 - \tfrac{1}{2}\, z_1 - \tfrac{1}{2} z_2 + \tfrac{1}{4}) = 2(z_1 - \tfrac{1}{2})(z_2 - \tfrac{1}{2}) \tag{52}$$

D. Approximation and Least Squares Fitting

In some cases, it may be desirable to find an approximate function of n multi-valued variables that fits a given complete function in accordance with some error criterion. The error criteria of this section is the method of least squares.

Let K be an index on the N rows of the function table (as before) and let y_K denote the actual value of the function evaluated for the values of the variables from the Kth row of the function table as in Equation 44. Let $\hat{y}_K$ denote the value of the approximating function evaluated for the values of the variables from the Kth row of the function table. The coefficients of the approximating polynomial are then to be chosen so that

$$E = \sum_{K=1}^{N} (y_K - \hat{y}_K)^2 \tag{53}$$

is a minimum.

The orthogonal p_i of Equations 41, 42, 43, and 47 are quite convenient in least squares approximation.

Theorem 3: Given any finite complete function of multi-valued variables expressed as shown in Equation 43, the approximate function formed by deleting one or more of the terms on the right-hand side of 43 is the least squares best fitting function in the remaining terms.

Proof: Equation 43 is rewritten here.

$$f\ (x_1,\ x_2,\ \ldots,\ x_n) = \sum_{m=1}^{T} c_m p_m \tag{43}$$

Since the order in which the terms on the right-hand side of Equation 43 are written is insignificant, assume only the first J terms on the right-hand side of Equation 43 are kept ($1 < J < T$), and the remaining terms are deleted. In accordance with Equation 45, the exact function values are given by

$$\begin{aligned} y_K &= \sum_{m=1}^{T} c_m p_{mK} \\ &= c_1 p_{1K} + c_2 p_{2K} + c_3 p_{3K} + \ldots + c_T p_{TK} \ . \end{aligned} \tag{45}$$

The approximate function values y_K can be written as

$$\begin{aligned} \hat{y}_K &= \sum_{m=1}^{J} c_m{}' \ p_{mK} \\ &= c_1{}' \ p_{1K} + c_2{}' \ p_{2K} + c_3{}' \ p_{3K} + \ldots + c_J{}' \ p_{JK} \end{aligned} \tag{54}$$

where $c_1{}',\ c_2{}',\ \ldots,\ c_J{}'$ are the coefficients which make the approximate function the least squares best fit to the complete function.

The squared error, which is greater than zero, is then given by

$$\begin{aligned} E &= \sum_{K=1}^{N} (y_K - \hat{y}_K)^2 \\ &= \sum_{K=1}^{N} y_K{}^2 - 2 \sum_{K=1}^{N} y_K \hat{y}_K + \sum_{K=1}^{N} \hat{y}_K{}^2 \\ &= \sum_{K=1}^{N} y_K{}^2 + \sum_{K=1}^{N} [-2\ y_K \sum_{m=1}^{J} c_m{}' \ p_{mK} + (\sum_{m=1}^{J} c_m{}' p_{mK})^2] \end{aligned} \tag{55}$$

Differentiating Equation 55 with respect to $C_L{}'$, $L=1,\ 2,\ \ldots,\ J$, and setting the result equal to zero yields J equations of the form

$$\frac{\partial E}{\partial c_L'} = \sum_{K=1}^{N} 2\left(-y_K p_{LK} + p_{LK} \sum_{m=1}^{J} c_m' \, p_{mK}\right) = 0 \tag{56}$$

or

$$\sum_{K=1}^{N} p_{LK} \sum_{m=1}^{J} c_m' \, p_{mK} = \sum_{K=1}^{N} y_K p_{LK} \quad . \tag{57}$$

Rearranging the summation on the left gives

$$\sum_{m=1}^{J} c_m' \sum_{K=1}^{N} p_{LK} p_{mK} = \sum_{K=1}^{N} y_K p_{LK} \quad . \tag{58}$$

Making use of Equations 41 and 42 gives

$$C_L' = \sum_{K=1}^{N} y_K p_{LK} \quad . \tag{59}$$

From Equation 47, we find

$$C_L' = C_L \tag{60}$$

Q.E.D.

Next, we will show that the error is given by

$$\begin{aligned} E &= c_{J+1}^{\;2} + c_{J+2}^{\;2} + \ldots + c_N^{\;2} \\ &= \sum_{K=1}^{N} y_K^{\;2} - c_1^{\;2} - c_2^{\;2} - \ldots - c_J^{\;2} \\ &= \sum_{K=1}^{N} y_K^{\;2} - \sum_{K=1}^{N} \hat{y}_K^{\;2} \quad . \end{aligned} \tag{61}$$

Squaring Equation 45 yields

$$y_K^{\;2} = (c_1 p_{1K} + c_2 p_{2K} + \ldots + c_T p_{TK})^2 \quad . \tag{62}$$

Carrying out the squaring indicated by E uation 62 and summing 62 over the rows of the function table gives

$$\sum_{K=1}^{N} y_K^{\ 2} = \sum_{K=1}^{N} (c_1 p_{1K} \sum_{m=1}^{T} c_m p_{mK} + c_2 p_{2K} \sum_{m=1}^{T} c_m p_{mK}$$

$$+ \dots + c_T p_{TK} \sum_{m=1}^{T} c_m p_{mK})$$

$$= \sum_{K=1}^{N} c_1 p_{1K} \sum_{m=1}^{T} c_m p_{mK} + \sum_{K=1}^{N} c_2 p_{2K} \sum_{m=1}^{T} c_m p_{mK}$$

$$+ \dots + \sum_{K=1}^{N} c_T p_{TK} \sum_{m=1}^{T} c_m p_{mK} \ . \qquad (63)$$

Rearranging the order of the summations on the right-hand side gives

$$\sum_{K=1}^{N} y_K^{\ 2} = \sum_{m=1}^{T} c_1 c_m \sum_{K=1}^{N} p_{1K} p_{mK} + \sum_{m=1}^{T} c_2 c_m \sum_{K=1}^{N} p_{2K} p_{mK}$$

$$+ \dots + \sum_{m=1}^{T} c_T c_m \sum_{K=1}^{N} p_{TK}\, p_{mK} \ . \qquad (64)$$

Making use of Equations 41 and 42 yields

$$\sum_{K=1}^{N} \hat{y}_K^{\ 2} = c_1^{\ 2} + c_2^{\ 2} + \dots + c_T^{\ 2} \ . \qquad (65)$$

Similarly, we write y_K in accordance with the results of Theorem 3 as

$$\hat{y}_K = c_1\, p_{1K} + c_2 p_{2K} + \dots c_J p_{JK} \ . \qquad (66)$$

Using the same procedures as those indicated in Equations 62 through 65, we find

$$\sum_{K=1}^{N} \hat{y}_K^{\ 2} = c_1^{\ 2} + c_2^{\ 2} + \dots + c_J^{\ 2} \ . \qquad (67)$$

The squared error is then given by

$$E = \sum_{K=1}^{N} (y_K - \hat{y}_K)^2$$

$$= \sum_{K=1}^{N} (c_1 p_{1K} + c_2 p_{2K} + \dots + c_T p_{TK} - c_1 p_{1K} - c_2 p_{2K} - \dots - c_J p_{JK})^2$$

$$= \sum_{K=1}^{N} (c_{J+1} p_{J+1K} + c_{J+2} p_{J+2K} + \dots + c_T p_{TK})^2 \quad . \tag{68}$$

If the procedures indicated from Equations 63 through 65 are carried out on Equation 68, the result is Equation 61

$$E = c_{J+1}^2 + c_{J+2}^2 + \dots + c_T^2$$

$$= \sum_{K=1}^{N} y_K^2 - c_1^2 - c_2^2 - \dots - c_J^2$$

$$= \sum_{K=1}^{N} y_K^2 - \sum_{K=1}^{N} \hat{y}_K^2 \quad . \tag{61}$$

Theorem 3 shows that a complete function written in the form of Equation 43 can be approximated by a least squares best fit by dropping one or more of the terms of Equation 43. The squared error is given by Equation 61. A good approximating function would be one that made the ratio R $(0 < R < 1)$ of Equation 69 small.

$$R = \frac{E}{\sum_{K=1}^{N} y_K^2} \quad . \tag{69}$$

E. Incomplete Functions

An incomplete function of multi-valued variables is not defined for all possible combinations of the variables. Table 3 shows a function of two three-valued z variables for which only six of the nine possible combinations of the variables produce a defined value of the function.

Table 3. An incomplete function of z_1 and z_2

z_2	z_1	$f(z_1, z_2)$
0	0	y_1
0	2	y_3
1	0	y_4
1	2	y_6
2	0	y_7
2	2	y_9

There are an infinite number of polynomials in z_1 and z_2 that will represent the function of Table 2, each giving a different set of values to the undefined points. One convenient choice would be to define the function as being zero at the previously undefined points.

However, consider Table 4 where all possible combinations of z_1 and z_2 are presented and where undefined values of the function are represented by u's in the function value volumn.

Table 4. A complete table of an incomplete function

z_2	z_1	$f(z_1, z_2)$
0	0	y_1
0	1	u_2 (undefined)

Table 4 (Continued)

z_2	z_1	$f(z_1, z_2)$
0	2	y_3
1	0	y_4
1	1	u_5 (undefined)
1	2	y_6
2	0	y_7
2	1	u_8 (undefined)
2	2	y_9

A polynomial representing the incomplete function can be found either through the method of Theorem 1 or by working with orthogonal terms of the p_i type using Equations 41 and 45. The polynomial that results is

$$
\begin{aligned}
f(z_1, z_2) &= y_1 + (-\tfrac{3}{2} y_1 + 2u_2 - \tfrac{1}{2} y_3)\, z_1 + (-\tfrac{3}{2} y_1 + 2y_4 - \tfrac{1}{2} y_7) z_2 \\
&+ (\tfrac{1}{2} y_1 - u_2 + \tfrac{1}{2} y_3)\, z_1^2 + (\tfrac{1}{2} y_1 - y_4 + \tfrac{1}{2} y_7)\, z_2^2 \\
&+ (\tfrac{9}{4} y_1 - 3u_2 + \tfrac{3}{4} y_3 - 3y_4 + 4u_5 - y_6 + \tfrac{3}{4} y_7 - u_8 + \tfrac{1}{4} y_9)\, z_1 z_2 \\
&+ (-\tfrac{3}{4} y_1 + \tfrac{3}{2} u_2 - \tfrac{3}{4} y_3 + y_4 - 2u_5 + y_6 - \tfrac{1}{4} y_7 + \tfrac{1}{2} u_8 - \tfrac{1}{4} y_9)\, z_1^2 z_2 \\
&+ (-\tfrac{3}{4} y_1 + u_2 - \tfrac{1}{4} y_3 + \tfrac{3}{2} y_4 - 2u_5 + \tfrac{1}{2} y_6 - \tfrac{3}{4} y_7 + u_8 - \tfrac{1}{4} y_9)\, z_1 z_2^2 \\
&+ (\tfrac{1}{4} y_1 - \tfrac{1}{2} u_2 + \tfrac{1}{4} y_3 - \tfrac{1}{2} y_4 + u_5 - \tfrac{1}{2} y_6 + \tfrac{1}{4} y_7 - \tfrac{1}{2} u_8 + \tfrac{1}{4} y_9)\, z_1^2 z_2^2 \qquad (70)
\end{aligned}
$$

It is noted that u_2, u_5, and u_8 may be arbitrarily chosen to make the coefficients of some of the terms equal to zero. Elimination of terms from the polynomial representing the function may be advantageous in certain applications. As an elementary example, the choice of

$$u_2 = \frac{1}{2} y_1 + \frac{1}{2} y_3 \qquad (71)$$

eliminates the z_1^2 term from Equation 70.

In addition, the polynomial which is the least squares best fitting approximation of an incomplete function may be of interest. Since the function may be defined in an infinite number of ways, in general, at the undefined points, orthogonal terms are of no particular aid in finding the least squares best fit. Generally, no polynomial exists which represents the function exactly and becomes a least squares best fit in the terms remaining after some of the terms are dropped.

However, the following approach will yield the least squares best fit for approximating an incomplete function. Let the approximate value of the function for the Kth row of the function table be represented by y_K and let m_i denote the terms which are functions of x_j that are retained in the approximating polynomial. Then

$$y_K = c_1 m_{1K} + c_2 m_{2K} + \dots + c_n m_{nK} \qquad (72)$$

where the c's are constants and m_{iK} is the evaluation of the m_i term for the values of the variables from the Kth row of the function table. The c's are chosen so that the polynomial

$$p = c_1 m_1 + c_2 m_2 + \dots + c_n m_n \qquad (73)$$

is the least squares best fit to the incomplete function. Then, if y_K is the exact value of the function for the Kth row of the function table and N is the number of rows in the incomplete function table, the c's are chosen to minimize

$$E = \sum_{K=1}^{N} (y_K - \hat{y}_K)^2$$

$$= \sum_{K=1}^{N} (y_K - c_1 m_{1K} - c_2 m_{2K} - \cdots - c_n m_{nK})^2 \qquad (74)$$

Differentiating Equation 74 with respect to C_L, $L = 1, 2, \ldots, n$, and setting the results equal to zero yields n equations in n unknowns of the form

$$c_1 \sum_{K=1}^{N} m_{1K} m_{LK} + c_2 \sum_{K=1}^{N} m_{2K} m_{LK} + \cdots + c_n \sum_{K=1}^{N} m_{nK} m_{LK}$$

$$= \sum_{K=1}^{N} y_K m_{LK} \quad . \qquad (75)$$

The system of equations indicated by Equation 75 can usually be solved to give the desired least squares polynomial coefficients of Equation 73. As the number of terms in Equation 73 increases, digital computer solutions of the system of equations becomes the only practical means to find the coefficients. Thus, finding a least squares best fit for an incomplete function is generally a much harder task than finding the least squares best fit of a complete function.

III. LOGIC WITH TERNARY VARIABLES

A. Polynomials Representing Ternary Devices

This section presents representations of functions of ternary variables in terms of real polynomials. Function tables that either represent or could represent ternary devices are presented and the corresponding real polynomial representations are given.

The first types of ternary devices considered are shown in Figures 1 and 2. The devices of these figures are single input devices where the input is represented as being a z_j or t_j variable. Electrically speaking, this means that the input can be only one of three distinct electrical states. The states might be three distinct voltage levels, three distinct current levels, three distinct phases of some signal compared with a reference signal, etc. Furthermore, the output can be only one of three distinct electrical states. The function that the device performs on the input is placed inside the boxes of Figures 1 and 2.

The three distinct electrical states may be associated with the three values of the z_j variable or with the three values of the t_j variable. The output is some function of input z_j variables and is arithmetically one more than the output function in terms of the t_j variables.

MacKay and MacIntyre (7) present a ternary counter circuit. Their basic ternary counter circuit is shown in Figure 3. The waveforms associated with the circuit are shown in Figure 4. The ternary counter may be represented by real polynomial in the following manner. Let one input state represent 0, 3, 6, 9, ... input pulses. Let a second input state

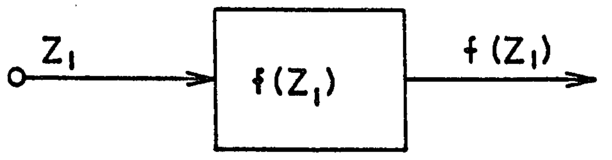

Figure 1. Representation of a single-input ternary device in terms of z variables

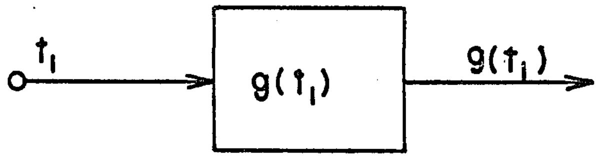

Figure 2. Representation of a single-input ternary device in terms of t variables

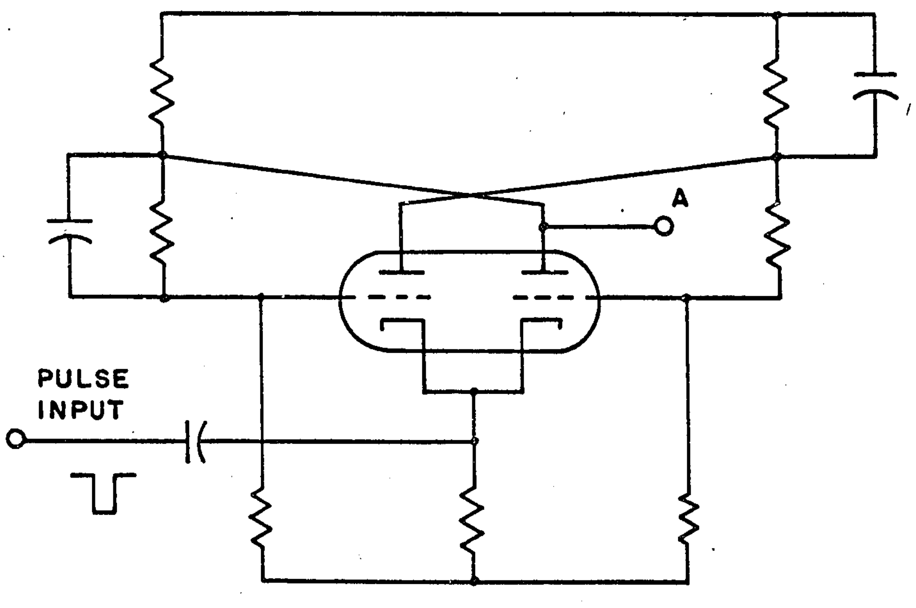

Figure 3. Ternary counter circuit

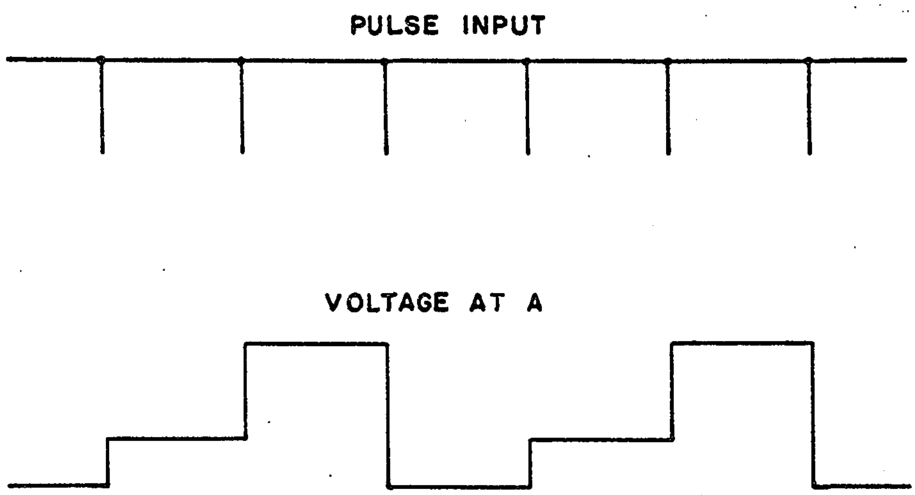

Figure 4. Waveforms of ternary counter circuit

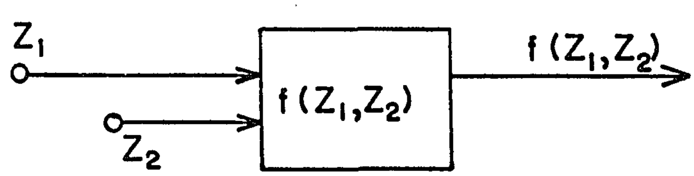

Figure 5. Representation of a two-input ternary device in terms of z variables

represent 1, 4, 7, 10, ... input pulses. Finally, let a third input state represent 2, 5, 8, 11, ... input pulses. Associate an output state with each of the three distinct voltage levels which appear at A of Figure 3. Under these conventions, the device can be represented as shown in Table 5 where the logical operation performed on the input is termed forward step.

Table 5. Function table for logical operation termed forward step

z_1	t_1	$f_1(z_1)$	$g_i(t_1)$
0	-1	1	0
1	0	2	1
2	1	0	-1

The real polynomials representing the functions indicated by Table 5 are

$$f_1(z_1) = -\frac{3}{2} z_1^2 + \frac{5}{2} z_1 + 1 \tag{76}$$

and

$$g_1(t_1) = -\frac{3}{2} t_1^2 - \frac{1}{2} t_1 + 1 \tag{77}$$

A ternary device which performs a logical operation termed backward step on a single input can be represented as shown in Table 6.

Table 6. Function table for a logical operation termed backward step

z_1	t_1	$f_2(z_1)$	$g_2(t_1)$
0	-1	2	1
1	0	0	-1
2	1	1	0

The real polynomials representing the functions indicated by Table 6 are

$$f_2(z_1) = \frac{3}{2} z_1^2 - \frac{7}{2} z_1 + 2 \tag{78}$$

and

$$g_2(t_1) = \frac{3}{2} t_1^2 - \frac{1}{2} t_1 - 1 \tag{79}$$

Next, consider the two-input ternary device of Figure 5. These devices have the same properties as those in Figures 1 and 2 except that the devices operate on two inputs to produce the output.

As a first example, consider the circuit of Figure 6. Assume in Figure 6 that the zener diode has a breakdown voltage of B and that the inputs, W_1 and W_2, take on only the voltage values 0, .5B, and 1.5B. The output voltage W_3 corresponding to the nine combinations of input voltages is given in Table 7.

Table 7. Relation between W_1, W_2, and W_3 of Figure 5

W_2	W_1	W_3
0	0	0
0	.5B	0
0	1.5B	.5B
.5B	0	0
.5B	.5B	.5B
.5B	1.5B	.5B
1.5B	0	0
1.5B	.5B	.5B
1.5B	1.5B	1.5B

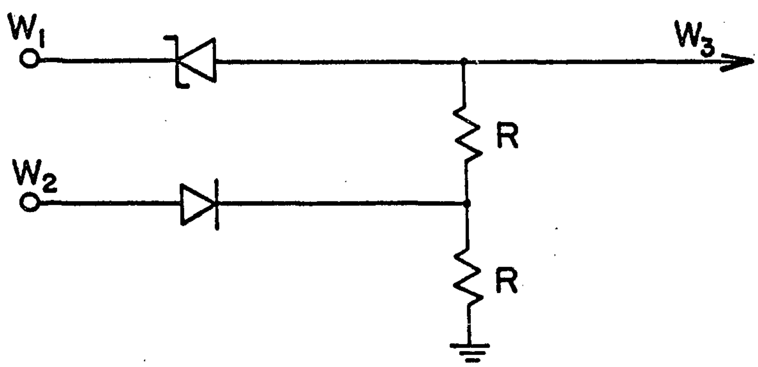

Figure 6. Ternary circuit with zener diode

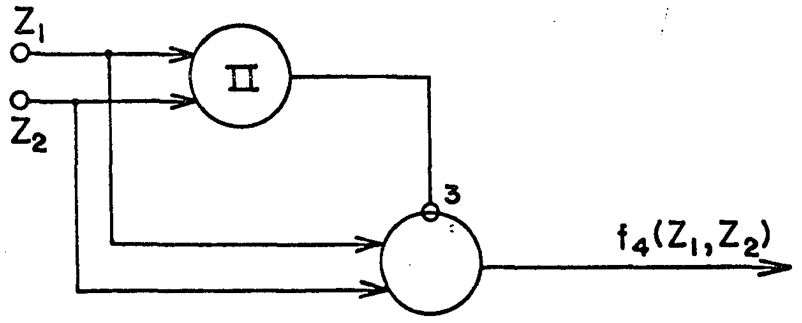

Figure 7. Modulo adder with two parametrons

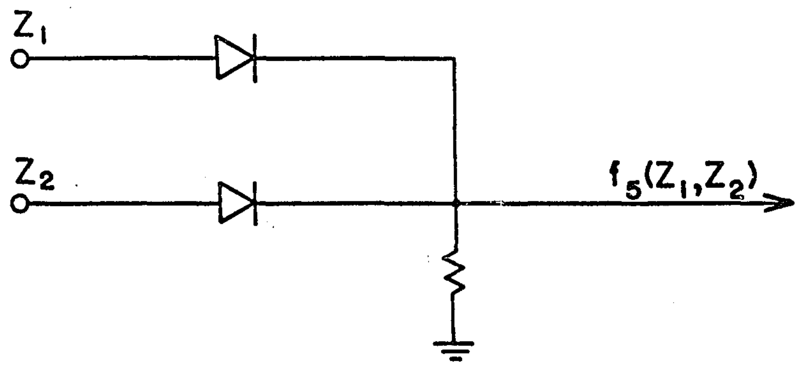

Figure 8. Circuit performing quasi-multiplication

The circuit of Figure 6 may be represented logically with z_j variables in which a voltage value of zero is associated with the value of zero of the z_j variable, a voltage value of .5B is associated with the value one of the z_j variable, and a voltage value of 1.5B is associated with the value of two of the z_j variable. The logical relation in terms of z_j variables of the circuit of Figure 6 is given in Table 8.

Table 8. Function table illustrating the logic of the circuit of Figure 5

z_2	z_1	$f_3(z_1, z_2)$
0	0	0
0	1	0
0	2	1
1	0	0
1	1	1
1	2	1
2	0	0
2	1	1
2	2	2

The real polynomial representing the function in Table 8 is

$$f_3\ (z_1,z_2) = -\tfrac{1}{2}\,z_1 + \tfrac{1}{2}z_1^{\,2} + \tfrac{13}{4}\,z_1z_2 - \tfrac{7}{4}\,z_1^{\,2}z_2 - \tfrac{5}{4}\,z_1z_2^{\,2} + \tfrac{3}{4}\,z_1^{\,2}z_2^{\,2} \tag{80}$$

Next, consider the parametrons presented in the articles by Schauer, et al., (13) and Hanson (3). Using the notation of the Schauer article, the inputs to a parametron are represented in the following way:

"0" - no oscillation

"1" - an oscillation in phase with a reference

"2" - an oscillation 180° out of phase with the reference

Complementation is defined as a phase inversion of a signal. The complement of "0" is "0". The complement of "1" is "2". The complement of "2" is "1".

The parametron is represented as a large circle with the threshold of the parametron indicated by a Roman numeral inside the circle. If no Roman numeral is present, the threshold is assumed to be one. A small circle is drawn where an input line meets a parametron when the input is to be complemented. The inputs are assumed to have weight one unless a number by an input line indicates a different weight.

The output of a parametron will be "0" unless the magnitude of the number that results from subtracting the number of inputs with an in phase oscillation from the number of inputs with a 180 degrees out of phase oscillation equals or exceeds the parametron threshold. In the latter case, the output of the parametron is a "1" when there are more in phase inputs than 180 degrees out of phase inputs and is a "2" when there are more 180 degrees out of phase inputs than in phase inputs.

A two-input ternary device comprised of two parametrons which performs the logical operation termed modulo addition is shown in Figure 7. The function table associated with the device is shown in Table 9.

Table 9. Function table for a modulo adder

z_2	z_1	$f_4(z_1, z_2)$
0	0	0

Table 9 (Continued)

z_2	z_1	$f_4(z_1,z_2)$
0	1	1
0	2	2
1	0	1
1	1	2
1	2	0
2	0	2
2	1	0
2	2	1

The real polynomial representing the function in Table 9 is

$$f_4(z_1,z_2) = z_1 + z_2 + \frac{21}{4} z_1z_2 - \frac{15}{4} z_1{}^2z_2 - \frac{15}{4}z_1z_2{}^2 + \frac{9}{4} z_1{}^2z_2{}^2 \tag{81}$$

Note that

$$f_4 (z_1, z_2) = f_4 (z_2, z_1) \tag{82}$$

A two-input ternary device which performs a logical operation termed quasi-multiplication is shown in Figure 8. The voltage levels associated with the z variables of Figure 8 are such that a zero represents the most positive voltage, a two represents the second most positive voltage, and a one represents the least positive voltage or zero voltage. The function table associated with the device of Figure 8 is given as Table 10.

Table 10. Function table for a logical operation termed quasi-multiplication

z_2	z_1	$f_5\ (z_1, z_2)$
0	0	0
0	1	0
0	2	0
1	0	0
1	1	1
1	2	2
2	0	0
2	1	2
2	2	2

The real polynomial representing the function in Table 10 is

$$f_5(z_1, z_2) = \tfrac{1}{2}\, z_1z_2 + \tfrac{1}{2}\, z_1^2z_2 + \tfrac{1}{2}\, z_1z_2^2 - \tfrac{1}{2}\, z_1^2z_2^2 \tag{83}$$

Note that

$$f_5\ (z_1, z_2) = f_5\ (z_2, z_1) \tag{84}$$

Next, consider the threshold device of Figure 9. The device has two input currents, represented by z_1 and z_2. When the sum of the two currents is great enough, the zener diode breaks down. One logical operation that could be performed by the device of Figure 9 is represented in Table 11 and is termed ternary half adder carry. The logical operation is such that the output function, $f_6\ (z_1, z_2)$, is zero except that in those cases where the sum of the two inputs, z_1 and z_2, is three or more, the output function is a one.

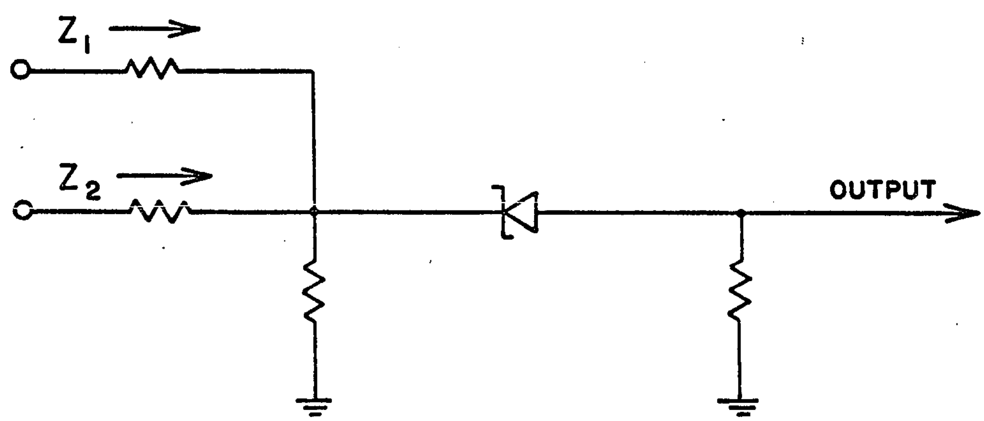

Figure 9. Threshold device with zener diode

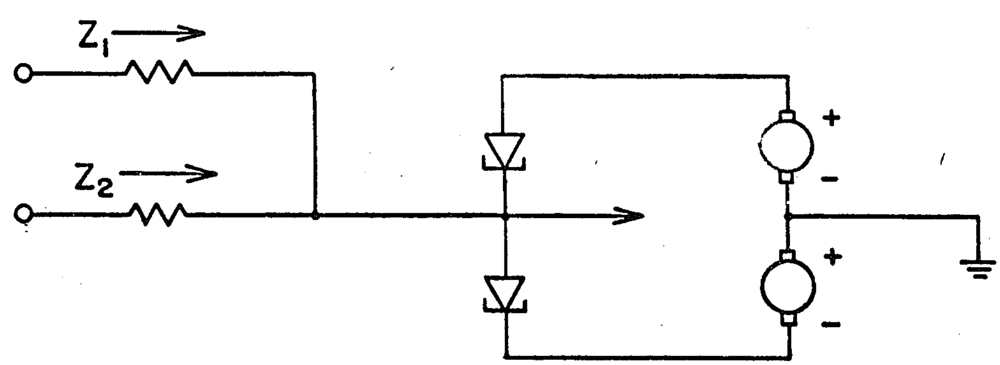

Figure 10. Goto pair circuit with two tunnel diodes

Table 11. Function table for a logical operation termed ternary half adder carry

z_2	z_1	$f_6\ (z_1,\ z_2)$
0	0	0
0	1	0
0	2	0
1	0	0
1	1	0
1	2	1
2	0	0
2	1	1
2	2	1

The real polynomial representing the function in Table 11 is

$$f_6\ (z_1,z_2) = -\frac{7}{4}\ z_1z_2 + \frac{5}{4}\ z_1z_2^{\ 2} + \frac{5}{4}\ z_1^{\ 2}z_2 - \frac{3}{4}\ z_1^{\ 2}z_2^{\ 2} \qquad (85)$$

Note that

$$f_6\ (z_1,\ z_2) = f_6\ (z_2,\ z_1) \qquad (86)$$

Another logical operation that could be performed by the device of Figure 9 is represented in Table 12. The output function, $f_7\ (z_1,z_2)$, of Figure 9 is zero except in the case where the sum of the two inputs, z_1 and z_2, is four, the output function is two.

Table 12. Function table representing a threshold device

z_2	z_2	$f_7(z_1,\ z_2)$
0	0	0

Table 12 (Continued)

z_2	z_2	$f_7\ (z_1, z_2)$
0	1	0
0	2	0
1	0	0
1	1	0
1	2	0
2	0	0
2	1	0
2	2	2

The real polynomial representing the function in Table 12 is

$$f_7\ (z_1, z_2) = \frac{1}{2}\ z_1 z_2 - \frac{1}{2}\ z_1^2 z_2 - \frac{1}{2}\ z_1 z_2^2 + \frac{1}{2}\ z_1^2 z_2^2 \qquad (87)$$

Note that

$$f_7\ (z_1,\ z_2) = f_7\ (z_2,\ z_1) \qquad (88)$$

Another example of a two-input ternary device is given by the "Goto-pair" circuit discussed in Sims, et al. (14) and shown in Figure 10. The Goto-pair circuit comprised of two tunnel diodes can be used as a majority logic device with three binary inputs. If two of the binary inputs are summed to produce one ternary input, the circuit of Figure 10 results. Table 13 shows the logic associated with the circuit of Figure 10.

Table 13. Function table representing the logic associated with the circuit of Figure 10

z_2	z_1	$f_8 (z_1, z_2)$
0	0	0
0	1	0
0	2	1
1	0	0
1	1	1
1	2	1

One real polynomial representation for the incomplete function of Table 13 is

$$f_8 (z_1,z_2) = - \tfrac{1}{2} z_1 + \tfrac{1}{2} z_1^2 + 2z_1z_2 - z_1^2z_2 \tag{89}$$

Lowenschuss (6) demonstrates a device made from two Rutz (11) transistors which is illustrated in Figure 11 and is a two-input ternary device. The logic associated with this device is given in Table 14.

Table 14. Function table representing the logic associated with the circuit of Figure 11

z_2	z_1	$f_9 (z_1, z_2)$	$f_{10} (z_1, z_2)$
0	0	0	0
0	1	1	0
0	2	0	2
1	0	1	0
1	1	0	2

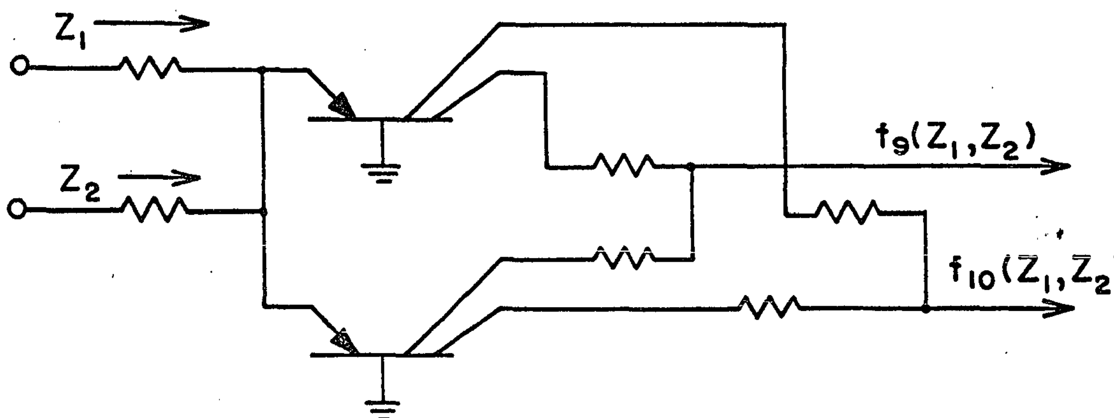

Figure 11. Ternary device with two Rutz transistors

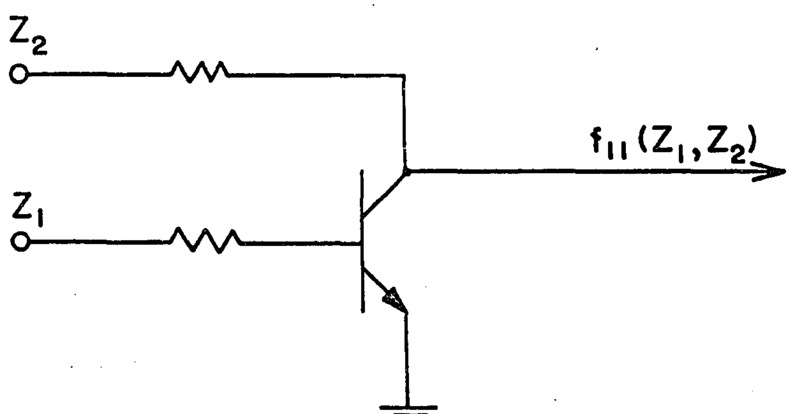

Figure 12. Gating device

Table 14 (Continued)

z_2	z_1	$f_9(z_1, z_2)$	$f_{10}(z_1, z_2)$
1	2	2	1
2	0	0	2
2	1	2	1
2	2	2	2

The real polynomials representing the functions in Table 14 are

$$f_9(z_1, z_2) = 2z_1 + 2z_2 - z_1^2 - z_2^2 - \frac{19}{2} z_1z_2 + 5z_1^2z_2 + 5z_1z_2^2 - \frac{5}{2} z_1^2z_2^2 \qquad (90)$$

and

$$f_{10}(z_1,z_2) = -z_1 - z_2 + z_1^2 + z_2^2 + \frac{19}{2} z_1z_2 - 5z_1^2z_2 - 5z_1z_2^2 + \frac{5}{2} z_1^2z_2^2 \qquad (91)$$

Note that

$$f_9(z_1, z_2) = f_9(z_2, z_1) \qquad (92)$$

and

$$f_{10}(z_1,z_2) = f_{10}(z_2, z_1) \qquad (93)$$

Finally, consider the two-input ternary device of Figure 12. In the circuit of Figure 12, if z_1 is a one or a two, the emitter-base diode of the transistor is biased off, the collector current is zero, and the output, $f_{11}(z_1,z_2)$, is the same as z_2. If z_1 is a zero, the transistor is biased on and the design parameters of the circuit can cause the output to be essentially zero. Table 15 shows the logic associated with the circuit of Figure 11 which may be regarded as a gating device.

Table 15. Function table for a gating device

z_1	$f_{11}(z_1, z_2)$
0	0
1	z_2
2	z_2

The real polynomial representing the function of Table 15 is

$$f_{11}(z_1, z_2) = (\tfrac{3}{2} z_1 - \tfrac{1}{2} z_1^2) z_2 \quad (94)$$

A full ternary adder is an example of a three-input ternary device. Designs for full ternary adders utilizing parametrons are given in Schauer (13) and Hanson (3). The logic associated with a full ternary adder is given in Table 16 where $f_{12}(z_1, z_2)$ is the "sum" associated with a full ternary adder and $f_{13}(z_1, z_2)$ is the "carry" associated with a full ternary adder.

Table 16. Function table representing a full ternary adder

z_3	z_2	z_1	$f_{12}(z_1, z_2, z_3)$	$f_{13}(z_1, z_2, z_3)$
0	0	0	0	0
0	0	1	1	0
0	0	2	2	0
0	1	0	1	0
0	1	1	2	0
0	1	2	0	1

Table 16 (Continued)

z_3	z_2	z_1	$f_{12}(z_1, z_2, z_3)$	$f_{13}(z_1, z_2, z_3)$
0	2	0	2	0
0	2	1	0	1
0	2	2	1	1
1	0	0	1	0
1	0	1	2	0
1	0	2	0	1
1	1	0	2	0
1	1	1	0	1
1	1	2	1	1
1	2	0	0	1
1	2	1	1	1
1	2	2	2	1
2	0	0	2	0
2	0	1	0	1
2	0	2	1	1
2	1	0	0	1
2	1	1	1	1
2	1	2	2	1
2	2	0	1	1
2	2	1	2	1
2	2	2	0	2

The real polynomials representing the functions of Table 16 are

$$f_{12}\ (z_1,z_2,z_3) = z_1 + z_2 + z_3 + \frac{21}{4}\ (z_1z_2 + z_1z_3 + z_2z_3)$$

$$- \frac{15}{4}\ (z_1{}^2z_2 + z_1{}^2z_3 + z_1z_2{}^2 + z_2{}^2x_3 + z_1z_3{}^2 + z_2z_3{}^2)$$

$$- \frac{287}{8}z_1z_2z_3 + \frac{9}{4}\ (z_1{}^2z_2{}^2 + z_1{}^2z_3{}^2 + z_2{}^2z_3{}^2)$$

$$+ \frac{135}{8}\ (z_1{}^2z_2z_3 + z_1z_2{}^2z_3 + z_1z_2z_3{}^2)$$

$$- \frac{63}{8}\ (z_1{}^2z_2{}^2z_3 + z_1{}^2z_2z_3{}^2 + z_1z_2{}^2z_3{}^2) + \frac{27}{8}\ z_1{}^2z_2{}^2z_3{}^2 \qquad (95)$$

and

$$f_{13}\ (z_1,z_2,z_3) = -\frac{7}{4}\ (z_1z_2 + z_1z_3 + z_2z_3)$$

$$+ \frac{5}{4}\ (z_1{}^2z_2 + z_1{}^2z_3 + z_1z_2{}^2 + z_2{}^2z_3 + z_1z_3{}^2 + z_2z_3{}^2)$$

$$+ \frac{89}{8}\ z_1z_2z_3 - \frac{3}{4}\ (z_1{}^2z_2{}^2 + z_1{}^2z_3{}^2 + z_2{}^2z_3{}^2)$$

$$- \frac{45}{8}\ (z_1{}^2z_2z_3 + z_1z_2{}^2z_3 + z_1z_2z_3{}^2)$$

$$+ \frac{21}{8}\ (z_1{}^2z_2{}^2z_3 + z_1{}^2z_2z_3{}^2 + z_1z_2{}^2z_3{}^2) - \frac{9}{8}\ z_1{}^2z_2{}^2z_3{}^2 \qquad (96)$$

Note that

$$\begin{aligned} f_{12}\ (z_1,z_2,z_3) &= f_{12}\ (z_1,z_3,z_2) \\ &= f_{12}\ (z_2,\ z_1,z_3) \\ &= f_{12}\ (z_2,z_3,z_1) \\ &= f_{12}\ (z_3,z_1,z_2) \\ &= f_{12}\ (z_3,z_2,z_1) \end{aligned} \qquad (97)$$

and

$$\begin{aligned} f_{13}(z_1,z_2,z_3) &= f_{13}\ (z_1,z_3,z_2) \\ &= f_{13}\ (z_2,z_1,z_3) \end{aligned}$$

$$= f_{13}\ (z_2, z_3, z_1)$$
$$= f_{13}\ (z_3, z_1, z_2)$$
$$= f_{13}\ (z_3, z_2, z_1) \tag{98}$$

B. Real Polynomial Identities

This section presents real polynomial identities which will be used in proving the logical relations of the next section. The first identity, which may be developed from a function table, is

$$z_j^{\ n} = (2^{n-1} - 1)\ z_j^{\ 2} - (2^{n-1} - 2)\ z_j \tag{99}$$

where z_j is a three-valued variable and n is an integer greater than zero. The relation is easily proved by substitution of the values which z_j can take on, namely, 0, 1, and 2, on each side of Equation 99 and seeing that an identity results.

The next identity is

$$t_j^{\ 2n-1} = t_j \tag{100}$$

where n is an integer greater than zero. The identity is easily proved by substitution of the values which t_j can take on, namely -1, 0, and 1, on each side of Equation 100 and seeing that an identity results.

The last identity presented is

$$t_j^{\ 2n} = t_j^{\ 2} \tag{101}$$

where n is an integer greater than zero. This identity also is easily proved by substitution of the values which t_j can take on in each side of Equation 101 and seeing that an identity results.

As an example of the use of the identities, consider the square of Equation 82

$$[f_4\ (z_1,z_2)]^2 = [z_1 + z_2 + \frac{21}{4} z_1z_2 - \frac{15}{4} z_1z_2^2 - \frac{15}{4} z_1z_2$$

$$+ \frac{9}{4} z_1^2\ z_2^2]^2 \ . \qquad (102)$$

It can be seen that if the squaring operation is carried out on the right-hand side of Equation 102 the resulting polynomial will contain terms containing z_1 to the third and fourth powers. The third and fourth powers of z_1 and z_2 may be substituted for with expressions containing only first and second powers by use of Equation 99. The actual equation that results from use of the foregoing procedures is

$$[f_4\ (z_1,z_2)]^2 = z_1^2 + z_2^2 + \frac{65}{4} z_1z_2 - \frac{39}{4} z_1^2\ z_2$$

$$- \frac{39}{4} z_1z_2^2 + \frac{21}{4} z_1^2z_2^2 \ . \qquad (103)$$

The form of Equation 103 could have been deduced in light of Equations 99 and 102 as

$$[f_4\ (z_1,z_2)]^2 = z_1^2 + z_2^2 + Az_1z_2 + Bz_1^2z_2 + Cz_1z_2^2 + Dz_1^2z_2^2 \qquad (104)$$

where A, B, C, and D are undetermined constant coefficients. Substitution from rows of a function table representing the left-hand side of Equation 104 produces a set of linear equations which may be solved for A, B, C, and D. It is noted that if the coefficients of the first two terms of Equation 104 are not deduced as being unity, they may be represented with undetermined coefficients and solved for the same manner as A, B, C, and D.

Applying the procedures discussed in the foregoing, it may also be shown that

$$[f_5(z_1,z_2)]^2 = -3z_1z_2 + 3z_1^2z_2 + 3z_1z_2^2 - 2z_1^2z_2^2 \ . \qquad (105)$$

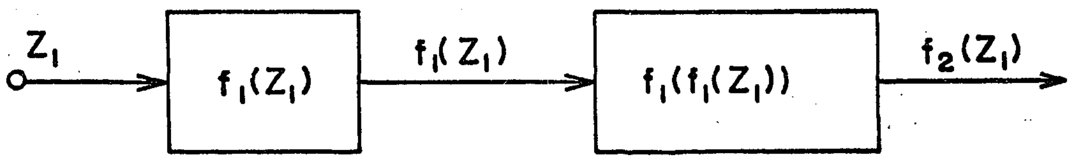

Figure 13. Schematic representation of a logical relation

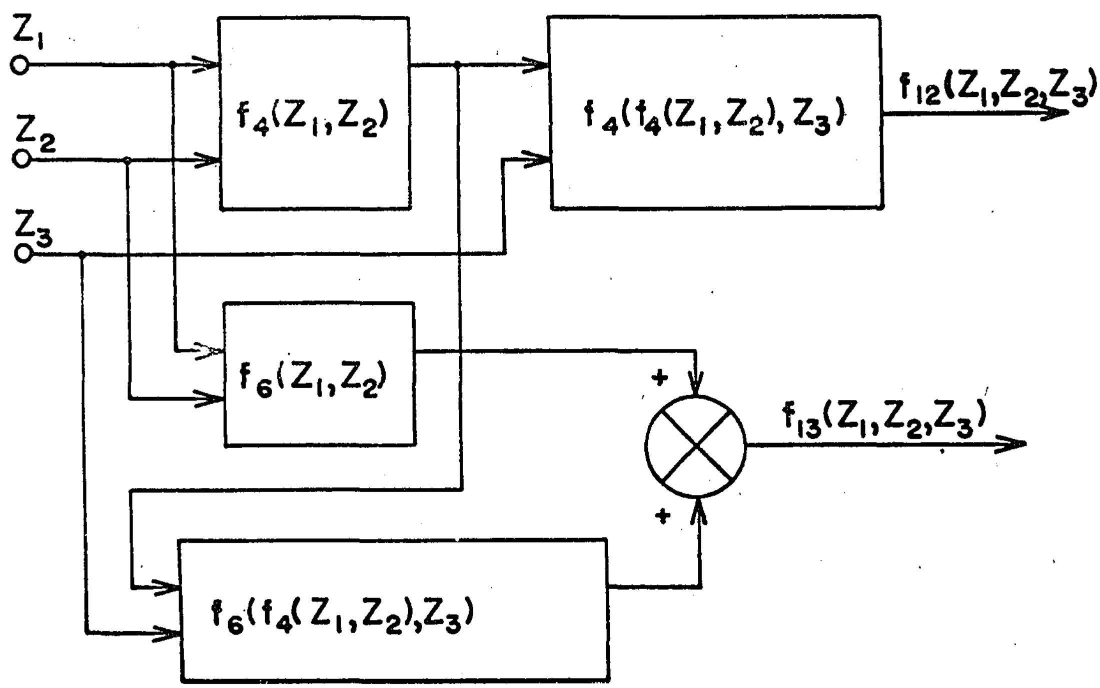

Figure 14. Schematic representation of a full ternary adder

C. Proofs of Logical Relations

This section will present some proofs of logical relations in terms of the real polynomials of the preceding section. The first proof will be that

$$f_2\ (z_1) = f_1\ (f_1(z_1))\quad . \tag{106}$$

The logical relation indicated by Equation 106 can be represented schematically as shown in Figure 13. The proof of Equation 106 proceeds by working with the right-hand side and using Equations 76, 78, and 99.

$$\begin{aligned} f_1\ (f_1(z_1)) &= -\tfrac{3}{2}\ [f_1(z_1)]^2 + \tfrac{5}{2}\ [f_1(z_1)] + 1 \\ &= -\tfrac{3}{2}\ [-\tfrac{3}{2}\ z_1{}^2 + \tfrac{5}{2}\ z_1 + 1]^2 + \tfrac{5}{2}\ [-\tfrac{3}{2}\ z_1{}^2 + \tfrac{5}{2}\ z_1 + 1] + 1 \\ &= -\tfrac{3}{2}\ [\tfrac{9}{4}\ z_1{}^4 - \tfrac{15}{2}\ z_1{}^3 + \tfrac{13}{4}\ z_1{}^2 + 5z_1 + 1] - \tfrac{15}{4}\ z_1{}^2 + \tfrac{25}{4}\ z_1 + \tfrac{5}{2} + 1 \\ &= -\tfrac{27}{8}\ z_1{}^4 + \tfrac{45}{4}\ z_1{}^3 - \tfrac{69}{8}\ z_1{}^2 - \tfrac{5}{4}\ z_1 + 2 \\ &= -\tfrac{27}{8}\ (7z_1{}^2 - 6z_1) + \tfrac{45}{4}\ (3z_1{}^2 - 2z_1) - \tfrac{69}{8}\ z_1{}^2 - \tfrac{5}{4}\ z_1 + 2 \\ &= \tfrac{3}{2}\ z_1{}^2 - \tfrac{7}{2}\ z_1 + 2 \\ &= f_2\ (z_1) \end{aligned} \tag{107}$$

Working with the t_1 variable, the relation equivalent to Equation 106 given by

$$g_2(t_1) = g_1\ (g_1(t_1)) \tag{108}$$

can be proved more easily. Working with the right-hand side of Equation 118 and making use of Equations 77, 79, 100, and 101 yields

$$\begin{aligned} g_1(t_1\ (t_1)) &= -\tfrac{3}{2}\ [g_1(t_1)]^2 - \tfrac{1}{2}\ [g_1(t_1)] + 1 \\ &= -\tfrac{3}{2}\ [-\tfrac{3}{2}\ t_1{}^2 - \tfrac{1}{2}\ t_1 + 1]^2 - \tfrac{1}{2}\ [-\tfrac{3}{2}\ t_1{}^2 - \tfrac{1}{2}\ t_1 + 1] + 1 \end{aligned}$$

$$= -\frac{3}{2}\left[\frac{9}{4}t_1^{\ 4} + \frac{3}{2}t_1^{\ 3} - \frac{11}{4}t_1^{\ 2} - t_1 + 1\right] + \frac{3}{4}t_1^{\ 2} + \frac{1}{4}t_1 - \frac{1}{2} + 1$$

$$= -\frac{27}{8}t_1^{\ 4} - \frac{9}{4}t_1^{\ 3} + \frac{39}{8}t_1^{\ 2} + \frac{7}{4}\ t_1 - 1$$

$$= -\frac{27}{8}t_1^{\ 2} - \frac{9}{4}\ t_1 + \frac{39}{8}t_1^{\ 2} + \frac{7}{4}\ t_1 - 1$$

$$= \frac{3}{2}t_1^{\ 2} - \frac{1}{2}t_1 - 1$$

$$= g_2\ (t_1)\ . \qquad (109)$$

Next, we shall prove the following relations.

$$f_{12}\ (z_1,\ z_2,\ z_3) = f_4\ (f_4\ (z_1,z_2),\ z_3) \qquad (110)$$

and

$$f_{13}\ (z_1,z_2,z_3) = f_6\ (z_1,z_2) + f_6\ (f_4(z_1,z_2),z_3) \qquad (111)$$

The logical relations of Equations 110 and 111 can be represented schematically as shown in Figure 14.

Figure 14 may be considered as representing what is termed a full ternary adder. The inputs z_1 and z_2 could represent the "old carry" that resulted from adding the next less significant digits of the two numbers. Then $f_{12}\ (z_1,z_2,z_3)$ is the digit of the same significance as z_1 or z_2 in the number of base three representing the sum of the two numbers being added, and $f_{13}\ (z_1,z_2,z_3)$ is the "new carry" that is added to the next more significant digits in the two numbers being added.

The proof of Equation 110 is found by working with the right-hand side and employing Equations 82, 95, 99, and 103.

$$f_4\ (f_4\ (z_1,z_2),z_3)$$
$$= f_4\ (z_1,z_2) + z_3 + \frac{21}{4}\ z_3\ f_4(z_1,z_2) - \frac{15}{4}\ z_3^{\ 2}\ f_4(z_1,z_2)$$

$$- \frac{15}{4} z_3[f_4 (z_1,z_2)]^2 + \frac{9}{4} z_3^2 [f_4 (z_1,z_2)]^2$$

$$= z_3 + (1 + \frac{21}{4} z_3 - \frac{15}{4} z_3^2) f_4 (z_1,z_2)+(- \frac{15}{4} z_3 + \frac{9}{4} z_3^2) [f_4 (z_1,z_2)]^2$$

$$= z_3 + (1 + \frac{21}{4} z_3 - \frac{15}{4} z_3^2)(z_1 + z_2 + \frac{21}{4} z_1z_2 - \frac{15}{4} z_1z_2^2 - \frac{15}{4} z_1^2z_2 + \frac{9}{4} z_1^2z_2^2)+(- \frac{15}{4} z_3 + \frac{9}{4} z_3^2)(z_1^2 + z_2^2 + \frac{65}{4} z_1z_2 - \frac{39}{4} z_1^2z_2 - \frac{39}{4} z_1z_2^2 + \frac{21}{4} z_1^2z_2^2)$$

$$= z_1 + z_2 + z_3 + \frac{21}{4} z_1z_2 + \frac{21}{4} z_1z_3 + \frac{21}{4} z_2z_3 - \frac{15}{4} z_1^2z_2 - \frac{15}{4} z_1^2z_3 - \frac{15}{4} z_1z_2^2 - \frac{15}{4} z_2^2z_3 - \frac{15}{4} z_1z_3^2 - \frac{15}{4} z_2z_3^2 - \frac{267}{8}z_1z_2z_3 + \frac{9}{4} z_1^2z_2^2 + \frac{9}{4} z_1^2z_3^2 + \frac{9}{4} z_2^2z_3^2 + \frac{135}{8} z_1^2z_2z_3 + \frac{135}{8} z_1z_2^2z_3 + \frac{135}{8}z_1z_2z_3^2 - \frac{63}{8} z_1^2z_2^2z_3 - \frac{63}{8} z_1^2z_2z_3^2 - \frac{63}{8}z_1z_2^2z_3^2 + \frac{27}{8} z_1^2z_2^2z_3^2$$

$$= f_{12} (z_1,z_2,z_3) \quad . \tag{112}$$

Similarly, the proof of Equation 111 is found by working with the right-side and employing Equations 82, 85, 98, and 103.

$$f_6 (z_1,z_2) + f_6 (f_4 (z_1,z_2), z_3)$$

$$= - \frac{7}{4} z_1z_2 + \frac{5}{4} z_1z_2^2 + \frac{5}{4} z_1^2z_2 - \frac{3}{4} z_1^2z_2^2 - \frac{7}{4} z_3 f_4 (z_1,z_2) + \frac{5}{4} z_3^2 f_4 (z_1,z_2) + \frac{5}{4} z_3 [f_4 (z_1,z_2)]^2 - \frac{3}{4} z_3^2[f_4(z_1,z_2)]^2$$

$$= - \frac{7}{4} z_1z_2 + \frac{5}{4} z_1z_2^2 + \frac{5}{4} z_1^2z_2 - \frac{3}{4} z_1^2z_2^2 + (- \frac{7}{4} z_3+ \frac{5}{4} z_3^2)$$

$$(z_1 + z_2 + \frac{21}{4} z_1z_2 - \frac{15}{4} z_1z_2^2 - \frac{15}{4} z_1^2z_2 + \frac{9}{4} z_1^2z_2^2)$$

$$+ (\frac{5}{4} z_3 - \frac{3}{4} z_3^2)(z_1^2 + z_2^2 + \frac{65}{4} z_1z_2 - \frac{39}{4} z_1^2z_2 - \frac{39}{4}z_1z_2^2$$

$$+ \frac{21}{4} z_1^2z_2^2$$

$$= - \frac{7}{4} z_1z_2 - \frac{7}{4} z_1z_3 - \frac{7}{4} z_2z_3 + \frac{5}{4} z_1^2z_2 + \frac{5}{4} z_1z_2^2 + \frac{5}{4} z_1z_2^2$$

$$+ \frac{5}{4} z_1^2z_3 + \frac{5}{4} z_1z_3^2 + \frac{5}{4} z_2^2z_3 + \frac{89}{8} z_1z_2z_3 - \frac{3}{4} z_1^2z_2^2$$

$$- \frac{3}{4} z_1^2z_3^2 - \frac{3}{4} z_2^2z_3^2 - \frac{45}{8} z_1^2z_2z_3 - \frac{45}{8} z_1z_2^2z_3 - \frac{45}{8}z_1z_2z_3^2$$

$$+ \frac{21}{8} z_1^2z_2^2z_3 + \frac{21}{8} z_1^2z_2z_3^2 + \frac{21}{8} z_1z_2^2z_3^2 - \frac{9}{8} z_1^2z_2^2z_3^2$$

$$= f_{13} (z_1,z_2,z_3) \quad . \tag{113}$$

D. Implementation of Product Terms

The polynomials of the preceding sections have terms containing products of the z_j variables. If these products could be implemented, then a weighted sum of the products thus formed would implement the polynomials directly. The weighted sum might theoretically be done in analog fashion. A great number of possible implementations for forming these products could be suggested. This section suggests implementation for forming the product of two z_j variables and three z_j variables.

Figure 15 shows a schematic representation of implementation for forming the product of two z_j variables, z_1 and z_2. The summing junction of Figure 15 might be implemented, for example, by analog summation of voltages or currents employing weighting resistors. The logical relation indicated by Figure 15 is that

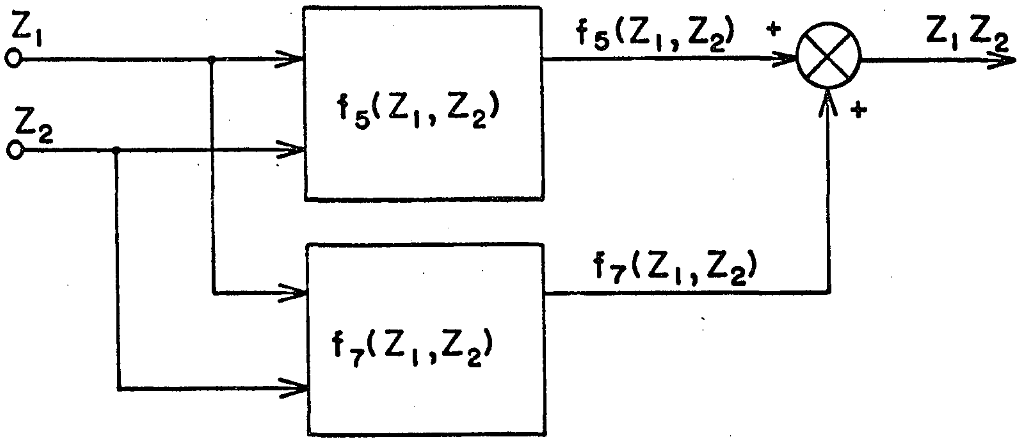

Figure 15. Schematic representation for implementation of $z_1 z_2$ product

$$z_1z_2 = f_5\ (z_1,\ z_2) + f_7\ (z_1,\ z_2) \quad . \tag{114}$$

Equation 114 can be proved by working with the right-hand side and making use of Equations 83 and 87.

$$\begin{aligned} &f_5\ (z_1,\ z_2) + f_7\ (z_1,\ z_2) \\ &\quad = \tfrac{1}{2}\ z_1z_2 + \tfrac{1}{2}\ z_1^2z_2 + \tfrac{1}{2}\ z_1z_2^2 - \tfrac{1}{2}\ z_1^2z_2^2 + \tfrac{1}{2}\ z_1z_2 - \tfrac{1}{2}\ z_1^2z_2 \\ &\qquad - \tfrac{1}{2}\ z_1z_2^2 + \tfrac{1}{2}\ z_1^2z_2^2 = z_1z_2 \end{aligned} \tag{115}$$

Figure 16 shows a schematic representation of implementation for forming the product of three z_j variables, z_1, z_2, and z_3. The logical relation indicated by Figure 16 is that

$$\begin{aligned} z_1z_2z_3 &= f_5\ (f_5\ (z_1,\ z_2),\ z_3) + f_{11}\ (z_3,\ f_7\ (z_1,\ z_2)) \\ &\quad + f_{11}\ (z_2,\ f_7(z_1,\ z_3)) + f_{11}\ (z_1,\ f_7\ (z_2,\ z_3)) \quad . \end{aligned} \tag{116}$$

Equation 116 can be proved by working with the right-hand side and making use of Equations 83, 87, 94, and 105.

$$\begin{aligned} &f_5\ (f_5\ (z_1,\ z_2),\ z_3) + f_{11}\ (z_3,\ f_7\ (z_1,\ z_2)) + f_{11}\ (z_2,\ f_7\ (z_1, z_3)) \\ &\quad + f_{11}\ (z_1,\ f_7\ (z_2,\ z_3)) \\ &= (\tfrac{1}{2}\ z_3 + \tfrac{1}{2}\ z_3^2)\ f_5\ (z_1,\ z_2) + (\tfrac{1}{2}\ z_3 - \tfrac{1}{2}\ z_3^2)\ [f_5\ (z_1,\ z_2)]^2 \\ &\quad + (\tfrac{3}{2}\ z_3 - \tfrac{1}{2}\ z_3^2)\ f_7\ (z_1,\ z_2) + (\tfrac{3}{2}\ z_2 - \tfrac{1}{2}\ z_2^2)\ f_7\ (z_1,\ z_3) \\ &\quad + (\tfrac{3}{2}\ z_1 - \tfrac{1}{2}\ z_1^2)\ f_7\ (z_2,\ z_3) \\ &= (\tfrac{1}{2}\ z_3 + \tfrac{1}{2}\ z_3^2)\ (\tfrac{1}{2}\ z_1z_2 + \tfrac{1}{2}\ z_1^2z_2 + \tfrac{1}{2}\ z_1z_2^2 - \tfrac{1}{2}\ z_1^2z_2^2) \\ &\quad + (\tfrac{1}{2}\ z_3 - \tfrac{1}{2}\ z_3^2)(-\ 3\ z_1z_2 + 3\ z_1^2z_2 + 3\ z_1z_2^2 - \tfrac{1}{2}\ z_1^2\ z_2^2) \\ &\quad + (\tfrac{3}{2}\ z_3 - \tfrac{1}{2}\ z_3^2)(\tfrac{1}{2}\ z_1z_2 - \tfrac{1}{2}\ z_1^2z_2 - \tfrac{1}{2}\ z_1z_2^2 + \tfrac{1}{2}\ z_1^2\ z_2^2) \end{aligned}$$

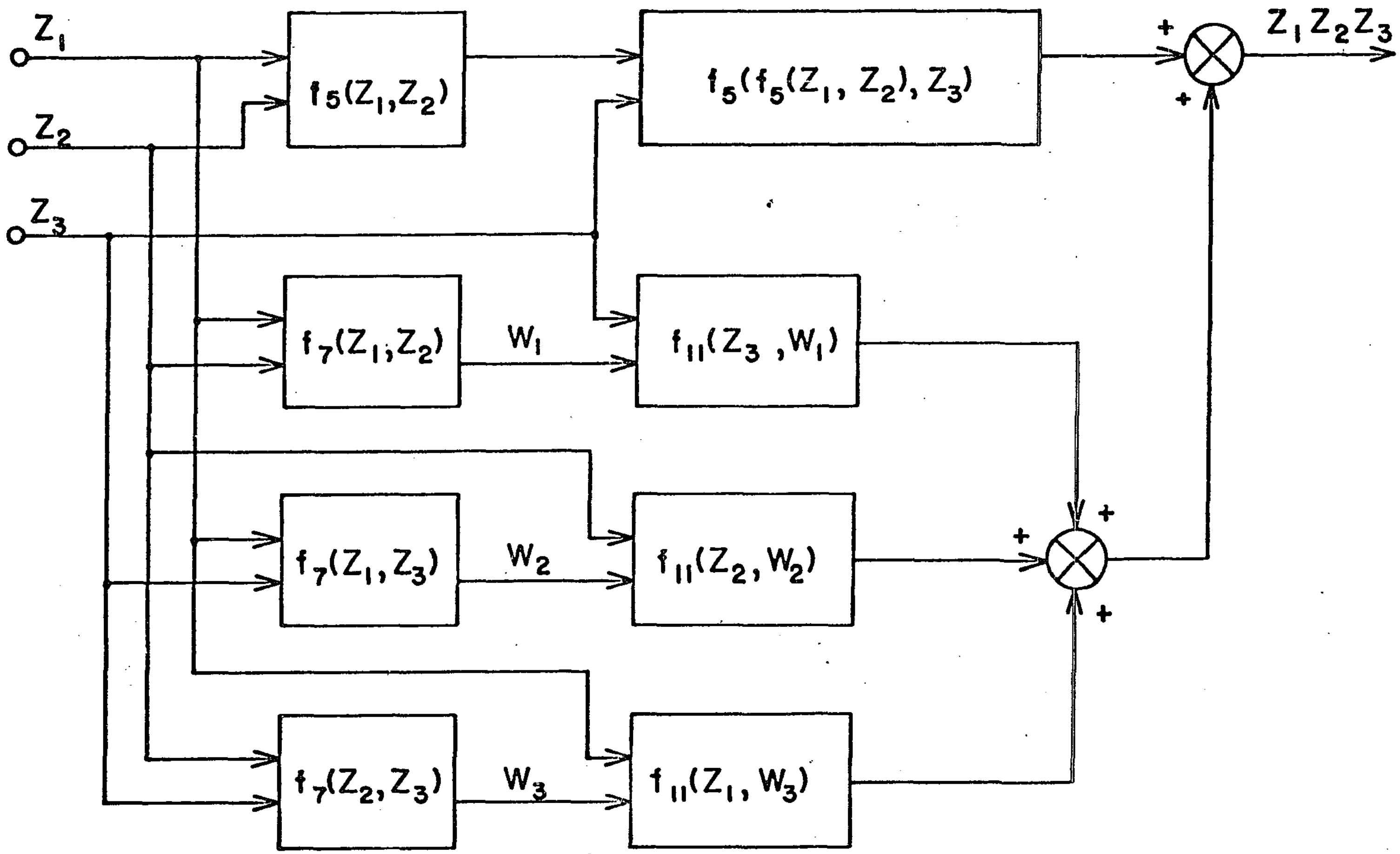

Figure 16. Schematic representation for implementation of $z_1 z_2 z_3$ product

$$+ \left(\frac{3}{2} z_2 - \frac{1}{2} z_2^2\right)\left(\frac{1}{2} z_1 z_3 - \frac{1}{2} z_1^2 z_3 - \frac{1}{2} z_1 z_3^2 + \frac{1}{2} z_1^2 z_3^2\right)$$

$$+ \left(\frac{3}{2} z_1 - \frac{1}{2} z_1^2\right)\left(\frac{1}{2} z_2 z_3 - \frac{1}{2} z_2^2 z_3 - \frac{1}{2} z_2 z_3^2 + \frac{1}{2} z_2^2 z_3^2\right)$$

$$= z_1 z_2 z_3 \quad . \tag{117}$$

IV. CODES AND FUNCTIONAL DECODING

A. Weighted Codes

A weighted code of two three-valued z variables is illustrated in Table 17.

Table 17. Function table for a weighted code

z_2	z_1	$f_{11}(z_1, z_2)$
0	0	0
0	1	1
0	2	2
1	0	3
1	1	4
1	2	5
2	0	6
2	1	7
2	2	8

The real polynomial representing the function of Table 17 is

$$f_{11}(z_1, z_2) = z_1 + 3 z_2 \quad . \tag{118}$$

More generally, a weighted code could be defined as being linear in the multi-valued variables and as being represented by the equation

$$f(x_1, x_2, \ldots, x_n) = b_o + b_1x_1 + b_2x_2 + \ldots + b_nx_n \tag{119}$$

where $b_o, b_1, \ldots, b_n$ are constants.

A single binary device has two well-defined states. A decimal device can be formed from four binary devices, since sixteen different conditions can be represented by the states of the four binary devices. Ten of the sixteen different conditions can be used to represent the integers 0, 1, 2, ..., 9 with the other six conditions not used or defined. If a decimal device were formed from two binary devices and one ternary device, there would be twelve different conditions. Ten of the twelve different conditions can be used to represent the integers 0, 1, 2, ..., 9 with only the other two conditions not used or defined.

Table 18 represents a weighted code which could represent a decimal device. In Table 18, z_1 and z_2 are two-valued variables and z_3 is a three-valued variable.

Table 18. Function table for a weighted code associated with a decimal device

z_3	z_2	z_1	$f_{15}(z_1,z_2,z_3)$
0	0	0	0
0	0	1	1
0	1	0	2
0	1	1	3
1	0	0	4
1	0	1	5
1	1	0	6
1	1	1	7
2	0	0	8
2	0	1	9

The real polynomial representing the function of Table 18 is

$$f_{15}(z_1, z_2, z_3) = z_1 + 2z_2 + 4z_3 \quad . \tag{120}$$

B. Non-weighted Codes

Just as weighted codes are useful, so are non-weighted codes. The real polynomial representing a non-weighted code is not linear in the multi-valued variables. Table 19 gives an example of a non-weighted code which is termed a reflected ternary code. This code possesses the property that only one z variable changes value for any two adjacent rows of the function table. Reflected codes find use in connection with analog-to-digital conversion devices.

Table 19. Function table for a reflected ternary code

z_2	z_1	$f_{16}(z_1, z_2)$
0	0	0
0	1	1
0	2	2
1	2	3
1	1	4
1	0	5
2	0	6
2	1	7
2	2	8

The real polynomial representing the function of Table 19 is

$$f_{16}(z_1, z_2) = z_1 + 7z_2 - 2z_2^2 - 4z_1z_2 + 2z_1z_2^2 \quad . \tag{121}$$

C. Functional Decoding

Real polynomials are useful in describing the decoding of a set of multi-valued variables into a function of the variables. Such descriptions find application in digital-to-analog conversion devices and can also be useful in devices which transform a digital input to a functionally related digital output.

As an example of functional decoding consider the "square" function of Table 20.

Table 20. Function table for a square function

z_2	z_1	$f_{17}(z_1, z_2)$
0	0	0
0	1	1
0	2	4
1	0	9
1	1	16
1	2	25
2	0	36
2	1	49
2	2	64

The real polynomial representing the function of Table 20 is

$$f_{17}(z_1, z_2) = z_1^2 + 9z_2^2 + 6z_1z_2 \quad . \tag{122}$$

D. Partitioning

For incomplete functions, partitioning the function table is a useful technique in finding a real polynomial representation. As previously noted, an incomplete function has an infinite number of real polynomial representations. Consider the square function of Table 21.

Table 21. Incomplete three variable square function

z_3	z_2	z_1	$f_{18}(z_1, z_2, z_3)$
0	0	0	0
0	0	1	1
0	0	2	4
0	1	0	9
0	1	1	16
0	1	2	25
0	2	0	36
0	2	1	49
0	2	2	64
1	0	0	81
1	0	1	100
1	0	2	121

The first nine rows of Table 21 have z_3 constant. The real polynomial which describes the first nine rows of Table 21 is independent of z_3 and is $f_{17}(z_1, z_2)$ given previously in Equation 122. The last three rows of Table 18 have z_2 and z_3 constant. The real polynomial which describes the last three rows of Table 18 is independent of z_2 and z_3 and is given by

$$f_{19}(z_1) = 81 + 18 z_1 + z_1^2 \quad . \tag{123}$$

A real polynomial that describes the incomplete function of Table 18 can be deduced as

$$f_{18}(z_1, z_2, z_3) = (1 - z_3)\, f_{17}(z_1, z_2) + z_3\, f_{19}(z_1)$$

$$= 81\, z_3 + z_1^2 + 9\, z_2^2 + 6\, z_1 z_2 + 18\, z_1 z_3 - 9\, z_2^2 z_3 + 6\, z_1 z_2 z_3 \,. \quad (124)$$

E. Segmented Approximation

The use of different polynomials to describe different parts of a given curve is termed segmented approximation. Partitioning a function table is a useful method for finding the segmented approximation of a given curve. It has been seen that a least squares best fitting approximation of a complete function is relatively easier to find than a least squares best fitting approximation of an incomplete function. The function table of an incomplete function can often be partitioned such that some of the partitions can be considered complete functions.

For example, the first nine rows of the incomplete function of Table 21 can be considered a complete function of the variables z_1 and z_2. A least squares best fitting approximation to the function describing the first nine rows may be found and may be used in the real polynomial describing all twelve rows according to the methods of the previous section.

In general, functions which are the least squares best fitting approximations to partitions of the function table may be found. These functions can then be combined to represent the entire function table according to the methods of the preceding section. In addition, it may be desirable to define undefined points of an incomplete function in order to simplify the finding of a good approximation to the function.

F. Interpolation

When a function of multi-valued discrete variables is a representation of a continuous function, interpolation is possible. For example, consider the square function of Table 20. The real polynomial representing the function is, as previously given,

$$f_{17}\ (z_1, z_2) = z_1^{\,2} + 9\ z_2^{\,2} + 6\ z_1 z_2 \quad . \tag{125}$$

Assume that z_2 is held constant at one of its three allowed values and the variable z_1 is allowed to vary continuously between zero and two rather than taking on its three allowed values only. With z_2 constant, $f_{17}\ (z_1, z_2)$ is a parabolic function in z_1. As z_1 varies continuously from zero to two, a continuous parabolic curve is described running through the three points where $f_{17}\ (z_1, z_2)$ was defined at z_1 equals zero, one, and two. The continuous square function which $f_{17}(z_1, z_2)$ is representing also varies continuously between the defined points where z_1 equals zero, one, and two. When well-behaved continuous functions are represented, interpolation can be used to give "finer grained" functions. This can be accomplished by replacing z_1 with

$$z_1 = \tfrac{1}{4}\ f_{14}\ (z_{11}, z_{12}) = \tfrac{1}{4}\ (z_{11} + 3\ z_{12}) \tag{126}$$

which places nine points on the continuous parabolic curve previously described rather than only three which z_1 itself would place on the curve.

V. CONCLUSIONS AND SUMMARY

The dissertation shows how to develop a real polynomial representation of a function of multi-valued variables from a function table. The least squares best-fitting approximation to a function is also discussed in terms of real polynomials.

Real polynomials are then presented which could represent ternary devices. The logic of networks containing, for the most part, ternary devices is demonstrated. Direct implementation of product terms of the real polynomials is considered and demonstrated for two special cases.

Weighted and non-weighted codes are presented. In particular, a weighted code with a mixture of two-valued and three-valued variables is presented.

Real polynomials which could be used in functional decoding are presented. Functional decoding finds use in digital-to-analog conversion devices and possibly in converting a digital input to a corresponding digital output. Segmented approximation of functions of multi-valued variables is discussed. Also discussed is interpolation for real polynomials which represent continuous functions.

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